Announcements

1) Last day to drop individual classes is today

2) HW #4 up, due next week

The Laplace Transform

(Section 7.2)

You may have seen the Fourier transform:

$$f(t) \xmapsto{\mathcal{F}} \int_{-\infty}^{\infty} f(t) e^{-ist} \, dt = \hat{f}(s)$$

Recall: improper integration!

If f is continuous, the improper integral of f is given by

$$\boxed{\int_a^\infty f(t)\,dt = \lim_{x \to \infty} \int_a^x f(t)\,dt}$$

where a is a real number and provided the limit exists!

Definition: (Laplace Transform)

If f is continuous on $[0,\infty)$, we define the **Laplace Transform** of f, denoted by $\mathcal{L}(f)$, as

$$\boxed{\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt}$$

for all $s \in \mathbb{R}$ such that the integral exists!

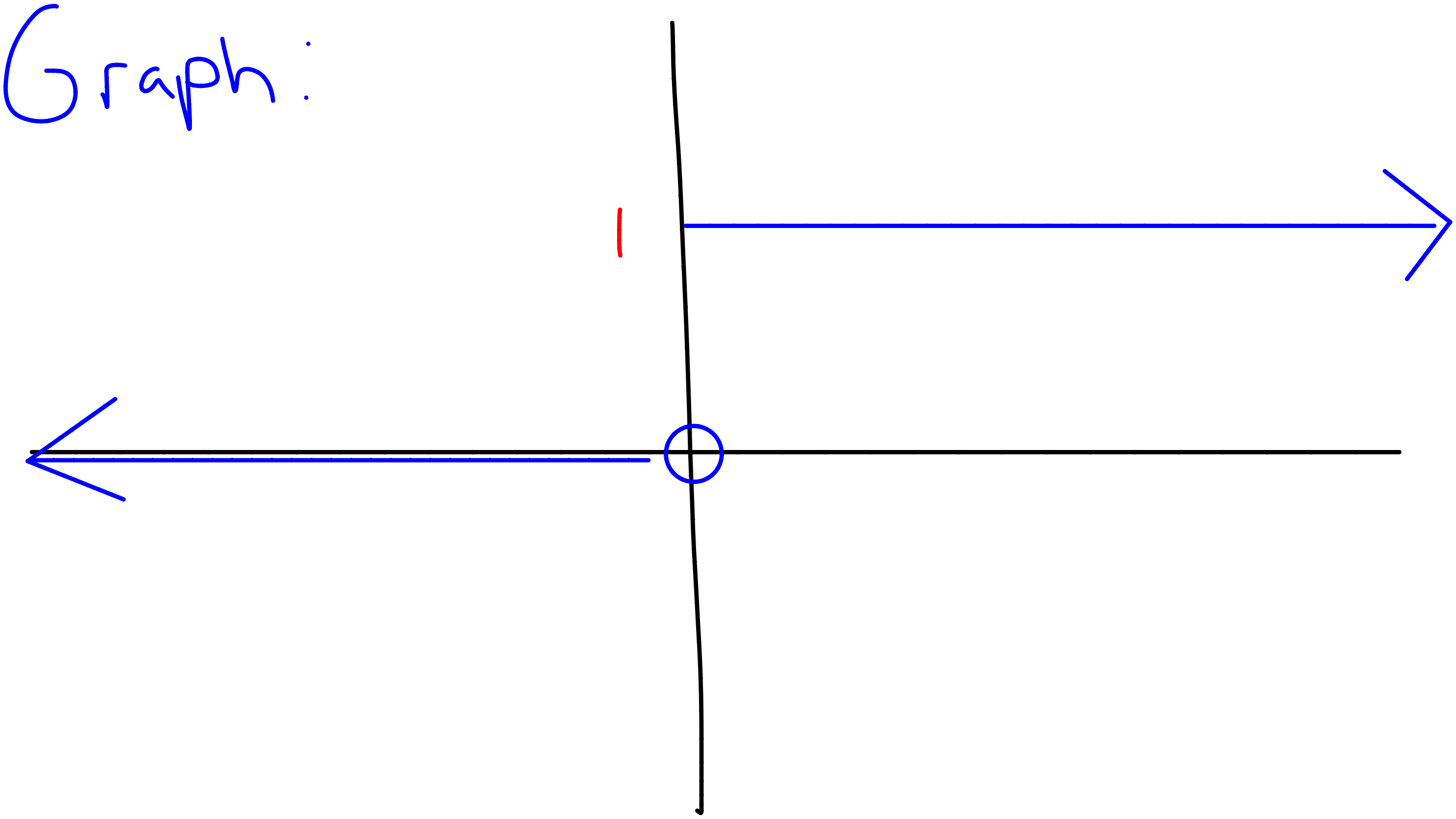
Moreover, we can take the Laplace Transform of functions with finitely many disconininuties because we can break up the integral over the points of discontinuity.

Example 1: (Heaviside Function)

Define the Heaviside Function

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Graph:



Compute $\mathcal{L}(u)$.

$$\mathcal{L}(u)(s) = \int_0^\infty u(t)e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

$$= \lim_{x\to\infty} \int_0^x e^{-st}\,dt$$

$$= \lim_{x\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^x$$

$(s \neq 0)$

$$= \lim_{x \to \infty} \left(\frac{e^{-sx}}{-s} + \frac{1}{s} \right)$$

$$= \lim_{x \to \infty} \left(-\frac{1}{se^{sx}} + \frac{1}{s} \right)$$

$$= \begin{cases} \frac{1}{s}, & s > 0 \\ \text{does not exist}, & s < 0 \end{cases}$$

If $s = 0$,

$$\mathcal{L}(f)(0) = \int_0^\infty 1 \, dt,$$

does not exist.

The domain is $s > 0$, and

on that domain,

$$\boxed{\mathcal{L}(u)(s) = \frac{1}{s}}$$

Continuous on its domain!

Example 2: Let $f(t) = t$.

Compute $\mathcal{L}(t)$

$$\mathcal{L}(t)(s) = \int_0^\infty t e^{-st}\, dt$$

$$= \lim_{x \to \infty} \int_0^x t e^{-st}\, dt$$

$s = 0$

$$\lim_{x \to \infty} \int_0^x t\, dt = \lim_{x \to \infty} \frac{t^2}{2} \Big|_0^x$$

$$= \lim_{x \to \infty} \frac{x^2}{2}$$

does not exist.

For $s \neq 0$,

$$\mathcal{L}(f)(s) = \lim_{x\to\infty} \underbrace{\int_0^x te^{-st}\,dt}_{\text{integrate by parts}}$$

$$u = t \qquad v = e^{-st}/-s$$

$$du = dt \qquad dv = e^{-st}\,dt$$

$$\int_0^x te^{-st}\,dt = \frac{te^{-st}}{-s} + \int_0^x \frac{e^{-st}}{s}\,dt$$

$$= \left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)\Bigg|_0^x$$

$$= \left(\frac{xe^{-sx}}{-s} - \frac{e^{-sx}}{s^2} \right) - \left(-\frac{1}{s^2} \right)$$

$$= \frac{-1}{e^{sx}} \left(\frac{x}{s} - \frac{1}{s^2} \right) + \frac{1}{s^2}$$

If $s > 0$,

$$\lim_{x \to \infty} \left(-\frac{1}{e^{sx}} \left(\frac{x}{s} - \frac{1}{s^2} \right) \right)$$

$$= 0$$ (l'Hopital's rule)

So if $s > 0$,

$$\mathcal{L}(t)(s) = \frac{1}{s^2}.$$

If $s < 0$

$$\lim_{x \to \infty} \left(\frac{-1}{e^{sx}} \left(\frac{x}{s} + \frac{1}{s^2} \right) \right)$$

does not exist.

The domain of $\mathcal{L}(t)$ is $s > 0$, and on that domain,

$$\boxed{\mathcal{L}(t)(s) = \frac{1}{s^2}}$$

Basic Properties

Given functions f and g such that $\mathcal{L}(f)$ and $\mathcal{L}(g)$ exist, then on the intersection of their domains,

1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

2) For any real number c,

$$\mathcal{L}(cf) = c\,\mathcal{L}(f)$$

These properties follow directly from the fact that they hold for integrals.

Not-so-basic Properties of the Laplace Transform

(Section 7.3)

1) $\mathcal{L}(e^{at}f)(s) = \mathcal{L}(f)(s-a)$

$$\mathcal{L}(e^{at}f)(s) = \int_0^\infty e^{at} f(t) e^{-st}\, dt$$

$$= \int_0^\infty f(t) e^{(a-s)t}\, dt$$

$$= \mathcal{L}(f)(s-a)$$

2) $\mathcal{L}(f')(s)$

$$= s\,\mathcal{L}(f)(s) - f(0)$$

Calculate:

$$\mathcal{L}(f')(s) = \int_0^\infty f'(t)e^{-st}\,dt$$

$$= \lim_{x\to\infty} \underbrace{\int_0^x f'(t)e^{-st}\,dt}_{\text{integrate by parts}}$$

$$\int_0^x f'(t)e^{-st}\,dt$$

$$u = e^{-st} \qquad v = f(t)$$

$$du = -se^{-st}\,dt \quad dv = f'(t)\,dt$$

$$\int_0^x f'(t)e^{-st}$$

$$= f(t)e^{-st}\Big|_0^x + \int_0^x s f(t)e^{-st}\,dt$$

$$= f(t)e^{-st}\Big|_0^x + s\int_0^x f(t)e^{-st}\,dt$$

So taking the limit as $x \to \infty$,

$$\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) + \lim_{x \to \infty} \frac{f(x)}{e^{sx}} - f(0)$$

$$= \boxed{s\,\mathcal{L}(f)(s) - f(0)}$$

provided $\displaystyle\lim_{x \to \infty} \frac{f(x)}{e^{sx}} = 0$!